Parallelizing Quantum Circuits

by Anne Broadbent
Université de Montréal
joint work with Elham Kashefi

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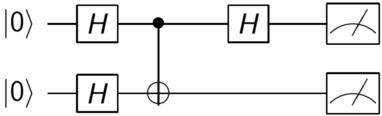
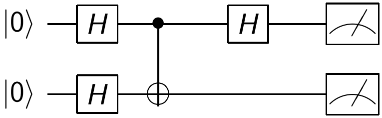


- *Size* of circuit: number of gates.

Quantum Circuits



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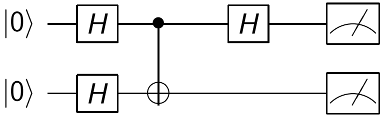

- *Size* of circuit: number of gates.
- *Depth* of circuit: length of longest input-output path, with 2-qubit gates acting as "bridges" between wires.
- (equivalently) the minimum number of layers required for the execution of the circuit, where each qubit interacts at most once in each layer.

Importance of the study of depth complexity

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- In computer science, one way to define classically *efficiently computable* problems is the class P which contains all problems computable in polynomial time. Another way to define *efficiently computable* is the class NC, which contains all classical polylogarithmic ($(\log n)^c$) depth circuits.

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- The power of parallel computers is thus characterized by NC, while P characterizes the power of sequential computers.
- NC $\subset$ P, but it is a long standing open question to determine whether or not P $=$ NC.
- We can define QNC and also ask whether or not P $\subset$ QNC or ultimately, BQP $=$ QNC.

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- For physical implementations, the number of steps required in the computation tells us how long the qubit must remain in a coherent state in order for the computation to have a good chance of success.

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$$\text{lower-depth} \Rightarrow \text{more computing power}$$

- Previous results obtained in the MBQC:
  - Clifford gates can be implemented in constant depth; so can Control-X and phase gates (R. Raussendorf, and H. J. Briegel 00)
  - Diagonal unitaries can be implemented in constant depth (D. E. Browne and H. J. Briegel 06)
  - Logarithmic classical depth is required for final corrections (Josza 05).

Result 1

All the models for QC are equivalent in computational power

Theorem

There exists a logarithmic separation in depth complexity between MBQC and the circuit model.

Result 2

An algorithm for parallelizing quantum circuits

1. Convert the circuit C to a pattern P.
2. Apply the tools of the measurement calculus to obtain a lower-depth pattern P'.
3. Convert P' back to a circuit, keeping all the auxiliary qubits.

Theorem

The above algorithm can decrease the depth of the circuit.

Result 3

Theorem

A pattern with flow $\mathcal{P}$ has depth $d+2$ if and only if on any influencing path we obtain $P^* N^{i \leq d} P^*$ $(\{Y, \emptyset\} N^{i \leq d} P^*)$, after applying the following rewriting rule:

$$N\, P_1^*\, \alpha_1 \beta_1\, P_2^*\, \alpha_2 \beta_2\, \cdots\, P_k^*\, N \begin{cases} NN & \textit{if} \quad \forall P_i^* \neq X(XY)^* \\ N & \textit{otherwise} \end{cases}$$

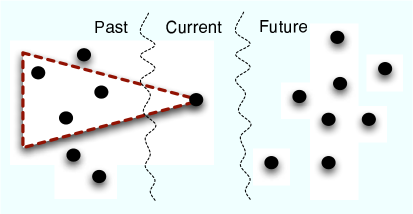
By understanding where the depth comes from, we can better understand how to design lower depth circuits and patterns.

Tools that we use

- measurement calculus
- standardization
- signal shifting
- flow
- influencing paths
- Pauli measurements

Depth

- Measurements and corrections depend on previous measurements; this is what creates depth in the measurement based model.



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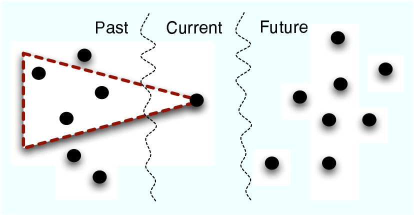

- Given qubit b, we give a method to list the qubits that b depends on. This is the "backward cone" of H. Briegel and R. Raussendorf.

Measurement Calculus

- V. Danos, E. Kashefi and P. Panangaden (2005) give us an algebraic formalism to reason about measurement patterns.

$$Z_k^s X_j^s \cdots {}_t[M_i^\alpha]^s \cdots E_G$$

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- *Dependencies*: $s = \sum_{i \in I} s_i$ is a *signal*; measurements and corrections can depend on signals. ${}_t[M_i^\alpha]^s$, means $M_i^{(-1)^s \alpha + t\pi}$.

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- To obtain parallel computation, we commute all the entanglement commands to the beginning.

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$${}_t[M_i^\alpha]^s X_i^r \Rightarrow {}_t[M_i^\alpha]^{s+r}$$
$${}_t[M_i^\alpha] Z_i^r \Rightarrow_{r+t} [M_i^\alpha]^s$$

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- Every pattern has a unique standard form.
- Could also be done using the biproduct method.

Standardization

Theorem

Standardization does not increase the depth of the pattern.

Signal Shifting

- A set of rules that say how to propagate the Z-dependency to the end.

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- If ${}_t[M_i^\alpha]^s$ appears in a pattern, replace it with $[M_i^\alpha]^s$.
- But for all future commands M, X or Z that have a dependency on s_i, we add dependency on t.
- The formal rules

$$
\begin{aligned}
{}_t[M_i^\alpha]^s &\Rightarrow S_i^t \; [M_i^\alpha]^s \\
X_j^s S_i^t &\Rightarrow S_i^t \; X_j^{s[(t+s_i)/s_i]} \\
Z_j^s S_i^t &\Rightarrow S_i^t \; Z_j^{s[(t+s_i)/s_i]} \\
{}_t[M_j^\alpha]^s S_i^r &\Rightarrow S_i^r \; {}_{t[(r+s_i)/s_i]}[M_j^\alpha]^{s[(r+s_i)/s_i]}
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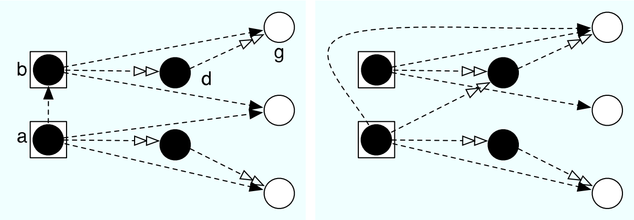

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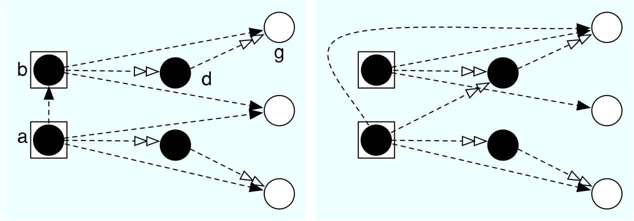

- Note: classical depth can increase

Classical depth

$$X_i^{s_1+s_2+\ldots+s_p}$$

The classical computation of the parity of the dependencies has depth in $\Theta(\log p)$.

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- It's reasonable to consider that classical computation is cheap compared to quantum computation.

Flow

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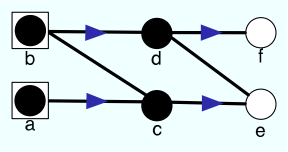
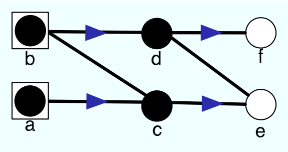


Flow

- Deterministic pattern: no matter what the measurements yield, the quantum output is always the same.
- The *flow* gives a recipe to construct the dependency structure.

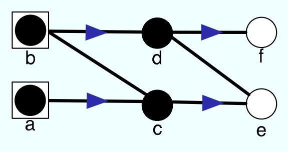


- From now on, we consider only patterns with flow (a sufficient condition for determinism, but not a necessary one!).

Influencing paths

We want to find the longest feed-forward path, which gives the depth. It's enough to consider only very specific paths in the graph:

- An influencing path: any input-output path in G that starts with a flow edge and has no two consecutive non-flow edges.



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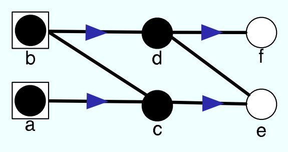

Theorem

If qubit b has a dependency on qubit a then a and b are on the same influencing path.

Pauli measurements and dependencies

- All Pauli measurements can be done in the first layer:

$$[M^X]^s = M^X$$
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- Pauli measurements also act as a "reset" for the depth.

Example 1

$$\ldots M_6^{\alpha} M_5^{X} M_4^{\alpha} M_3^{X} M_2^{\alpha} M_1^{X}$$

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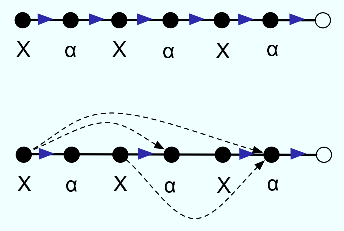

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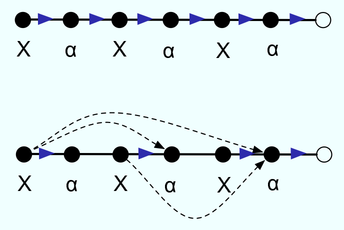
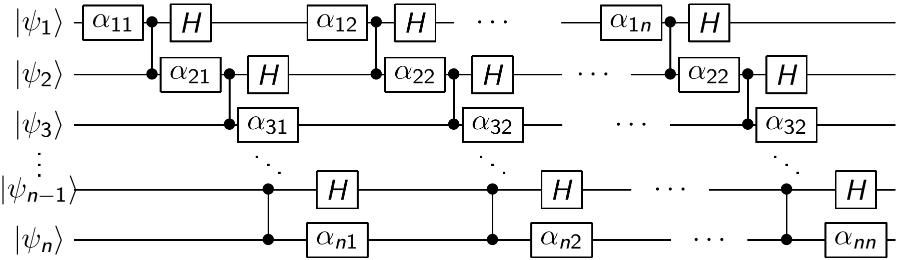


Depth = 2

Example 2



- This polynomial-depth circuit can be implemented in depth 2 in the measurement-based model.

Depth characterisation

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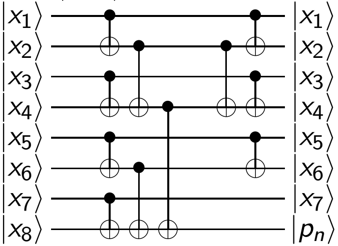

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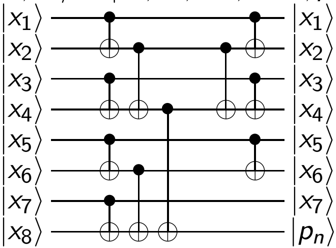

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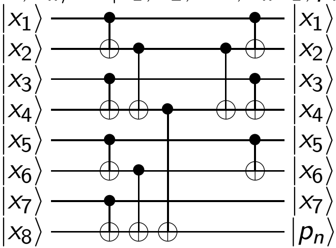

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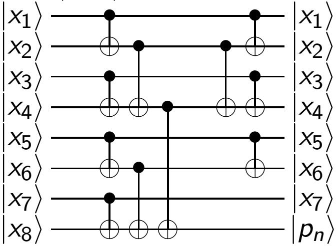

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- Proof: The state of the last output qubit depends on every input qubit; this requires depth $\log n$.
- But parity is in the Clifford group, so it has quantum depth 1 and classical depth in $O(\log n)$.

Conclusion

- We've shown how the tools of MBQC are useful in the design of parallel circuits.

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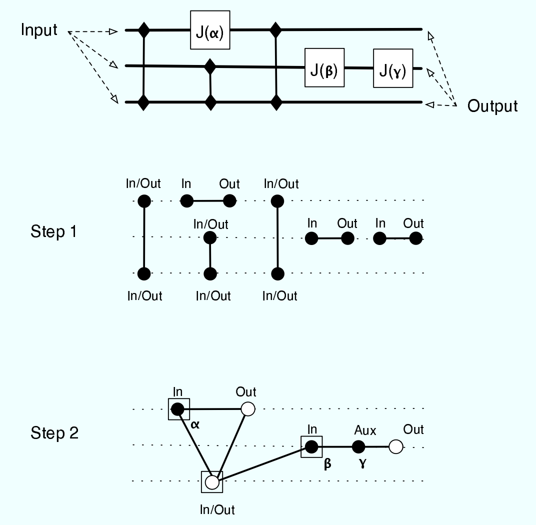
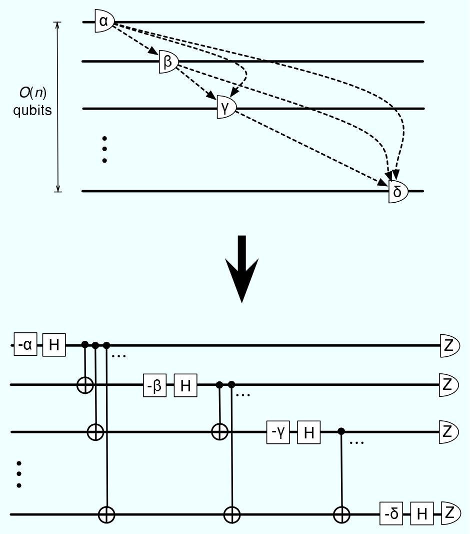
- We've characterized classes of patterns of depth d.
- The MBQC gives a clear separation between quantum and classical depth, as illustrated by the parity example.

Thank you!

From circuits to patterns

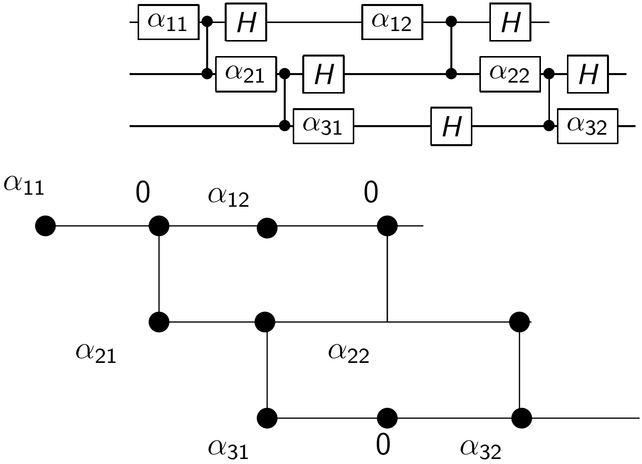


From patterns to circuits



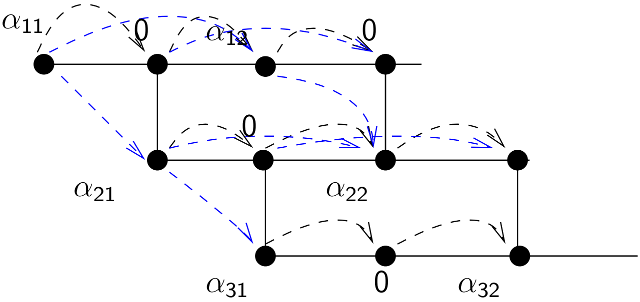
Example 2

- Convert circuit to pattern:



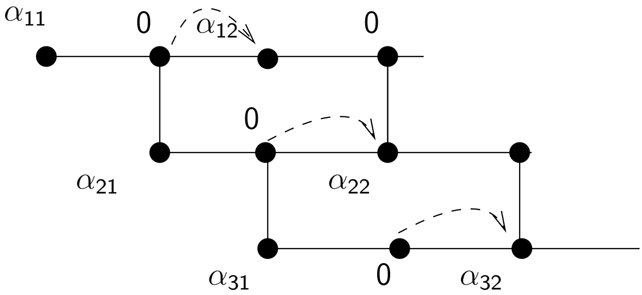
Example 2

- with dependencies



Example 2

- After pauli measurements and signal shifting



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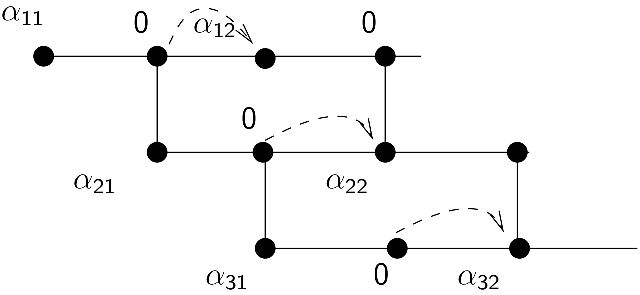

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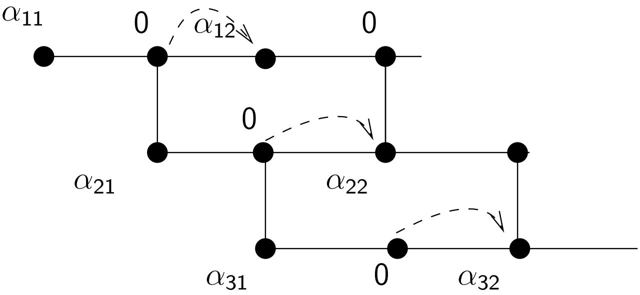

- The 0 measurement have no dependencies and can all be performed in the first layer. The second layer contains all the α measurements.
- Classical depth is $O(\log n)$.