Interpretation of Quantum Theory, Phys 490/773
Problem Set 3
Due: March 21st

1. *Some useful gaussian integrals:.*

a) Show that

$$\int_{-\infty}^{\infty} dx e^{-x^2/2\sigma^2} = \sqrt{2\pi\sigma^2}.$$

Hint: consider its square, $\int\int dxdy \exp -(x^2/2\sigma^2) \exp -(y^2/2\sigma^2)$, and change from rectangular coordinates to cylindrical coordinates.

b) Show that

$$\int_{-\infty}^{\infty} dx e^{-ax^2} e^{bx} = \sqrt{\pi/a} e^{-b^2/4a}.$$

Hint: make a linear change of integration variable so that you can use a).

c) Show that

$$\int_{-\infty}^{\infty} dx \frac{e^{-(x-a)^2/2\sigma_1^2}}{\sqrt{2\pi\sigma_1^2}} \frac{e^{-(x-a)^2/2\sigma_2^2}}{\sqrt{2\pi\sigma_2^2}} = \frac{e^{-(a-b)^2/2[\sigma_1^2+\sigma_2^2]}}{\sqrt{2\pi[\sigma_1^2+\sigma_2^2]}}$$

2) *Gambler's ruin game*.

In the gambler's ruin game, gambler 1 starts with $D_1(0)$ dollars and gambler 2 starts with $D_2(0)$ dollars, where $D_1(0) + D_2(0) = D$. After the nth coin toss, the ith gambler has a fraction $x_i(n) \equiv D_i(n)/D$ of the total money in the game. Show that the following two properties hold:

a) $x_1(n) + x_2(n) = 1$.

b) $\overline{x_i(n)} = x_i(0)$ (the Martingale property) where $\overline{x_i(n)}$ is the average value of $x_i(n)$ over the ensemble of all possible games. Hint: let $P_n^i(x)$ be the probability that $x_i = x$ after the nth toss, so $\sum_x P_n^i(x) = 1$, where the allowed values of x are 0, D^{-1}, $2D^{-1}$, ...$DD^{-1} = 1$. Explain why $P_n^i(x) = c_1 P_{n-1}^i(x - D^{-1}) + c_2 P_{n-1}^i(x + D^{-1})$, giving the values of the two constants. Now, $\overline{x_i(n)} = \sum_x x P_n^i(x)$. Use your difference equation to prove $\overline{x_i(n)} = \overline{x_i(n-1)}$, and use an induction argument.

The argument that the third property holds, $\overline{x_1(n) x_2(n)} \to 0$, i.e., that all games surely end, is rather complicated.

3) *CSL Lite*.

Consider the CSL Lite statevector:

$$|\psi, t\rangle_B = \sum_n \alpha_n |a_n\rangle e^{-(4\lambda t)^{-1}[B(t) - 2\lambda a_n t]^2},$$

where the probability of this statevector is $P_t(B) dB/\sqrt{2\pi\lambda t}$, with $P_t(B) \equiv_B \langle\psi, t|\psi, t\rangle_B$.

a) Define $x_n(t)$ as the squared magnitude of the *normalized* statevector $|\psi,t\rangle_B/P_t^{1/2}(B)$'s coefficient of $|a_n\rangle$. Show that $\sum_n x_n(t) = 1$.

b) Show that $\overline{x_n(t)} \equiv \int x_n(t)P_t(B)dB/\sqrt{2\pi\lambda t} = x_n(0)$.

c) Show that $\overline{x_n^{1/2}(t)x_m^{1/2}(t)} \to 0$ as $t \to \infty$, for $n \neq m$. Explain why the proof of collapse given in class, which depended upon $\overline{x_n(t)x_m(t)} \to 0$, is unaltered.

4) *True and False Collapse*.

Consider the statevector evolution

$$|\psi,t\rangle_B = \sum_n \alpha_n|a_n\rangle e^{iB(t)a_n},$$

which never collapses. $B(t)$ occurs with probability

$$P(B(t))dB(t) = \frac{e^{-(2\lambda t)^{-1}B^2(t)}}{\sqrt{2\pi\lambda t}}.$$

a) Construct the density matrix for this ensemble of statevectors. Hint: you will need the result of problem 1b).

b) Show that this "false collapse" density matrix is identical to the "true collapse" density matrix constructed from the statevector ensemble of problem 3.

5) *Free Particle Model*.

Consider the situation in which a free particle evolves according to the Hamiltonian $H = P^2/2m$, while also collapsing toward eigenstates of the position operator $A \equiv X$. The modified Schrödinger equation obeyed by the wavefunction is

$$\frac{\partial\langle x|\psi,t\rangle}{\partial t} = \frac{-i}{m}\frac{\partial^2\langle x|\psi,t\rangle}{\partial x^2} - \frac{1}{4\lambda}[w^2(t) - 2\lambda x]^2\langle x|\psi,t\rangle.$$

Choose a solution of the form of a gaussian wavefunction, $\langle x|\psi,t\rangle = \exp[-a(t)x^2 + b(t)x + c(t)]$.

a) Put this wavefunction into the modified Schrödinger equation and so obtain three equations for the three functions $a(t)$, $b(t)$ and $c(t)$.

b) Consider only the equation for $a(t)$. Solve it (be careful: $a(t)$ is complex!). Show that, for large t, $a(t)$ approaches a constant value, $(\lambda m/\hbar)^{1/2}(1-i)/2$.

c) If a free particle wavepacket of width s evolves under just the usual Schrödinger evolution, it expands during a short time Δt by $\Delta s \approx (\hbar/ms)\Delta t$. Explain this.

d) A wavepacket of width s, under just the collapse evolution, shrinks over Δt to $s[1 - \lambda s^2\Delta t]$. Use this and the result in c) to explain the result in b).