

# Decoherence in superconductors I

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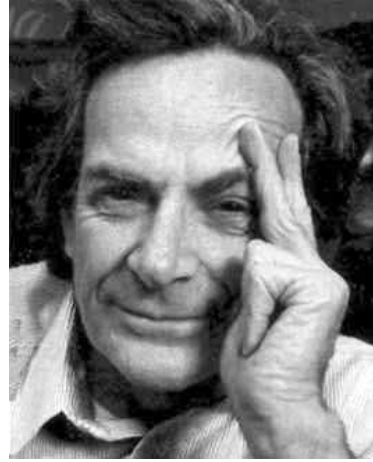
# Overview

- How to quantize classical friction?
- Quantizing a circuit with resistors
- Including nonlinear elements
- The Bloch-Redfield master equation
- Engineering decoherence

# The challenge of quantizing friction

- Dephasing is **irreversible**  
[Demonstration: Blackboard]
- Energy dissipation is **irreversible**
- Schrödinger's equation is **reversible** and conserves purity  
[Demonstration: Blackboard]
- Standard Lagrange formalism does not accommodate damping

# Quantum Brownian Motion



Caldeira

Leggett

Feynman

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) - \hat{q} \sum c_i \hat{x}_i + \sum \left( \frac{\hat{p}_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 \hat{x}_i^2 \right) + \hat{q}^2 \sum_i \frac{c_i^2}{2m_i \omega_i^2}$$

Integrate out oscillator bath: Blackboard

$$m\ddot{q} + \int_0^t dt' \gamma(t-t') \dot{q}(t') + \frac{\partial V}{\partial q} = \xi(t)$$

# Why oscillator baths?

Weak perturbation to any manybody quantum system

$$\hat{H} = \hat{H}_0 - \hat{F} \hat{Q} \text{ implies } \langle \delta Q(\omega) \rangle = \chi(\omega) F(\omega) \quad \chi(\omega) = i \langle Q(t > 0) Q(0) \rangle_{\omega, eq}$$

Complex decomposition  $\chi = \chi' + i\chi''$

$$\text{Fluctuations} \quad \frac{1}{2} \langle \delta Q(t) \delta Q(0) + \delta Q(0) \delta Q(t) \rangle_{\omega} = \chi'$$

$$\text{Dissipation} \quad \langle E(t) \rangle_{\omega} = \omega \chi'' |F|^2$$

The correlation function completely determines the response, higher moments are trivial: Oscillators are enough

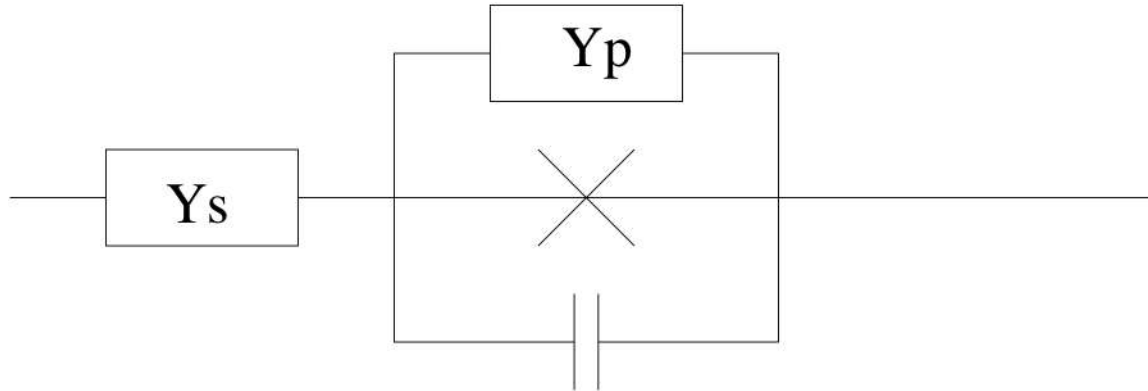
Central limit theorem: The distributions of (nondegenerate) random variables will be Gaussian in the thermodynamic limit

... does apply to Fermionic systems as well

... does not apply to some systems highlighted in lecture 3

# Computing J

Compute noise or friction – on phase or charge



[Analysis on the blackboard]

$$Y_p = 1/R : C \left( \frac{\Phi_0}{2\pi} \right) \ddot{\phi} + \frac{\Phi_0}{2\pi R} \dot{\phi} + I_c \sin \phi = I, J = \omega / R$$

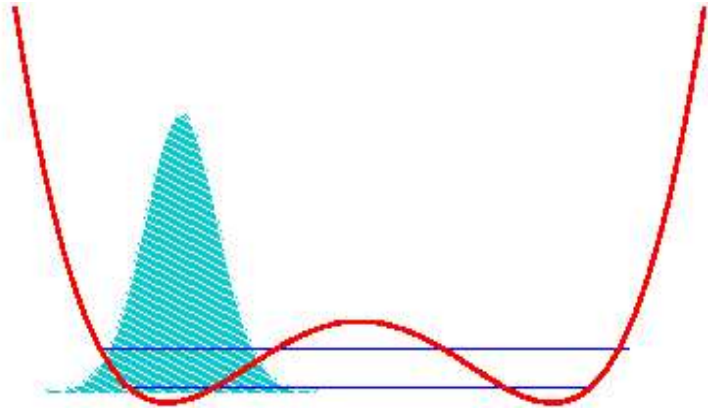
More general (Fourier space):

$$D(\omega) \phi = -\frac{\partial V}{\partial \phi}$$

$$D(\omega) = -\omega^2 \left( \frac{\Phi_0}{2\pi} \right)^2 + i \left( \frac{\Phi_0}{2\pi} \right) \omega Y(\omega)$$

$$J(\omega) = \Im(D(\omega))$$

# Discrete variables



Tight-binding (two-site) approximation

$$x = \pm q_0 = \sigma_z$$

$$J_{TLS}(\omega) = \sum \lambda_i^2 \delta(\omega - \omega_i) = \frac{q_0^2}{2\pi \hbar} J(\omega)$$

$$H = \frac{\epsilon}{2} \sigma_z + \frac{\Delta}{2} \sigma_x + \frac{1}{2} \sigma_z \sum \lambda_i (a_i + \bar{a}_i) + \sum \omega_i (\bar{a}_i a_i + 1/2) \quad X = \sum \lambda_i (a_i + \bar{a}_i)$$

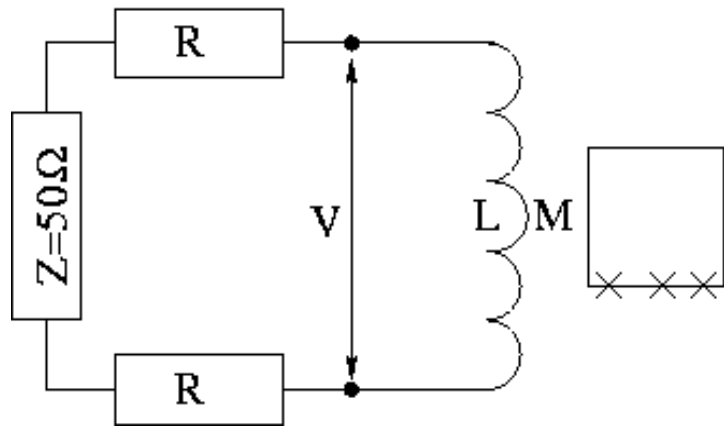
Related to quantum noise [Derivation: Blackboard]

$$J(t) = \langle X(t) X(0) \rangle = \int d\omega J(\omega) \left( \cos \omega t \coth \left( \frac{\omega}{2T} \right) - i \sin \omega t \right)$$

$$S(t) = \frac{1}{2} \langle X(t) X(0) + X(0) X(t) \rangle = \int d\omega J(\omega) \cos \omega t \coth \left( \frac{\omega}{2T} \right)$$

$$S(\omega) = J(\omega) \coth \left( \frac{\omega}{2T} \right)$$

# Characterization from noise



## Johnson-Nyquist-Noise

$$\frac{1}{2} \langle V(t)V(0) + V(0)V(t) \rangle_{\omega} = \hbar \omega \Re Z \coth\left(\frac{\omega}{2T}\right)$$

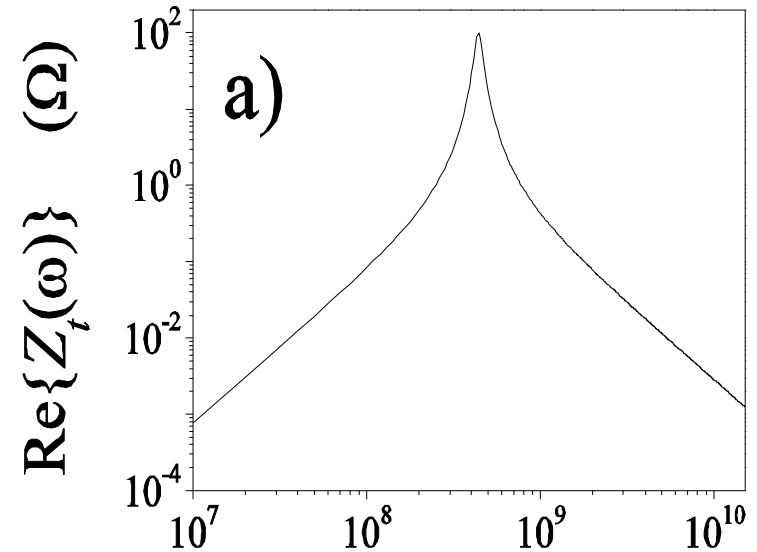
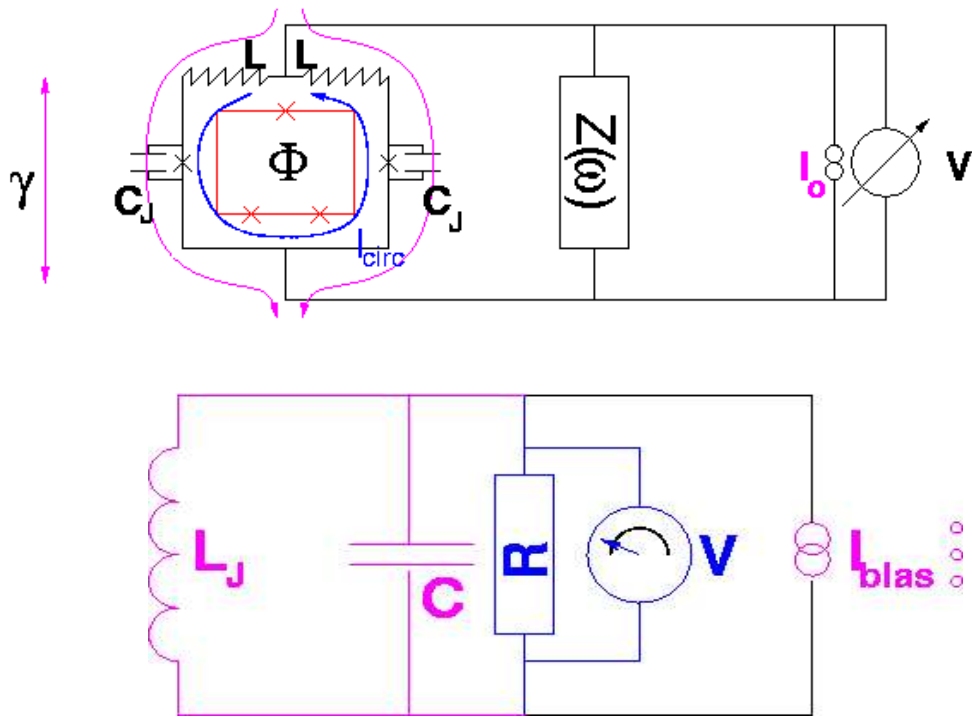
$$J(\omega) = \frac{\alpha \omega}{1 + \omega^2 / \omega_c^2}$$

$$\omega_c = (Z + 2R) / L$$

$$\alpha = \frac{4 M^2 I_p^2}{h(Z + 2R)}$$

- + physical connection to decoherence evident
- + works entirely in the two-state approximation
- difficult to apply to more complex circuits

# Linearization of nonlinear circuits

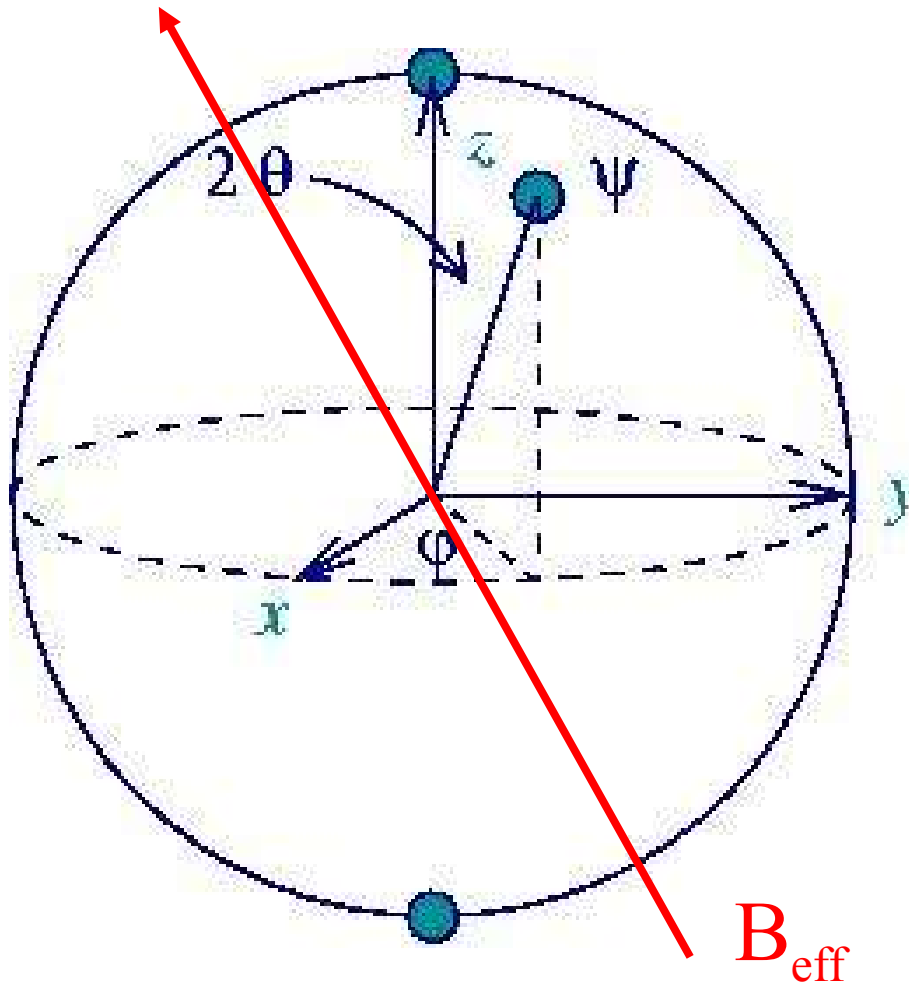


[Derivation: Blackboard]

$$J(\omega) = \frac{(2\pi)^2}{\hbar\omega} \left( \frac{MI_p}{\Phi_0} \right)^2 I_{bias}^2 \tan^2 \left( \frac{\pi\Phi}{\Phi_0} \right) \Re Z_{eff}$$

# The Bloch sphere

Convenient representation of the two-state Hamiltonian and state



$$\vec{S} = \begin{pmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \\ \langle \sigma_z \rangle \end{pmatrix}$$

$$H = \gamma \vec{B} \cdot \vec{\sigma}$$

# The Bloch equation

Simple phenomenology (NMR): Bloch equation

$$\dot{\vec{M}} = \gamma \vec{B} \times \vec{M} - \begin{pmatrix} M_x/T_2 \\ M_y/T_2 \\ (M_z - M_{z,eq})/T_1 \end{pmatrix} \quad \vec{B} = \begin{pmatrix} \delta B_x(t) \\ \delta B_y(t) \\ B_z \end{pmatrix}$$

M: Magnetization

$$M_i = N M_0 \langle \sigma_i \rangle$$

$T_1$ : Bit flip errors

$$|0\rangle \rightarrow |1\rangle$$

$T_2$ : Phase errors

$$|0\rangle + |1\rangle \rightarrow |0\rangle - |1\rangle$$



*Felix Bloch*

# Bloch equations and spectroscopy

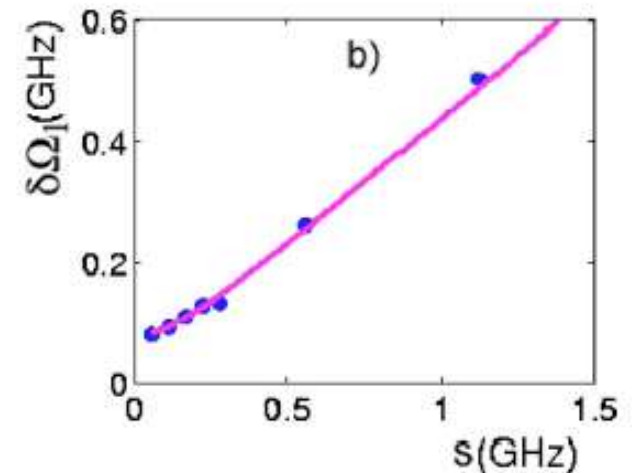
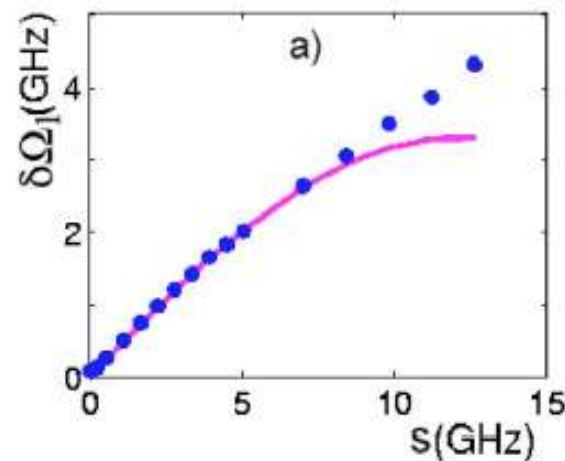
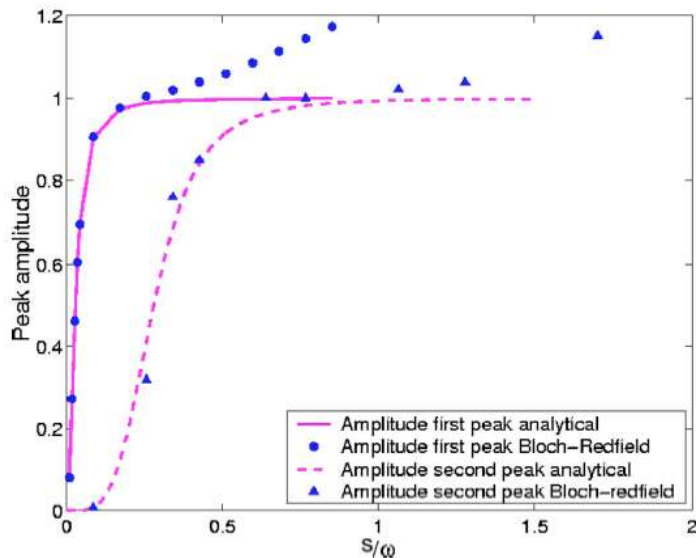
Energy splitting:  $\Delta E = gB_z$

Rotating drive:  $\delta B_x = (\hbar\omega_1/g)\cos\omega t$   $\delta B_y = (\hbar\omega_1/g)\sin\omega t$

Resonances at  $\omega = \Delta E$

Line width

$$\delta\omega = \sqrt{(\omega - \Delta E)^2 + \omega_1^2 T_1/T_2}$$



M.C. Goorden and F.K. Wilhelm  
Phys. Rev. B **68**, 012508 (2003)

# System-Bath model + RDM

System + Environment Hamiltonian  $H = H_S + H_{SB} + H_B$

Liouville equation:  $\dot{\rho} = -i[H, \rho] =: L\rho$       L: Liouville superoperator

Reduced density matrix  $\rho_R = Tr_B(\rho) =: P\rho$       P: Zwanzig projector

Irrelevant density matrix:  $\rho_I = (1-P)\rho$

EOM:  $\dot{\rho}_I = (1-P)L\rho = (1-P)L\rho_I + (1-P)L\rho_R$

Formal solution:  $\rho_I(t) = \int_0^t dt' e^{-(1-P)Lt'} (1-P)L\rho_R + e^{(1-P)Lt} \rho_I$

Factorized boundary conditions:

Last term disappears

$$\rho(t=0) = \rho_S \otimes \rho_B$$

# Nakajima –Zwanzig equation

Plugging back:  $\dot{\rho}_R(t) = PL\rho_R + \int_0^t dt' PL e^{-(1-P)Lt'} (1-P)L\rho_R$

Sort terms:  $L = L_S + L_B + L_{SB}$

$$\dot{\rho}_R(t) = P(L_S + L_{SB})\rho_R + \int_0^t dt' PL_{SB} e^{-(1-P)Lt'} (1-P)L_{SB}\rho_R$$

Nakajima-Zwanzig equation, exact (but complicated)!

**Approximation 1:** Second order in  $L_{SB}$  (Born)

$$\dot{\rho}_R(t) = P(L_S + L_{SB})\rho_R + \int_0^t dt' PL_{SB} e^{-(1-P)(L_S + L_B)t'} (1-P)L_{SB}\rho_R$$

**Problem:**

Integro-differential equation, nonlocal in time

# Markov approximation

$$\dot{\rho}_R(t) = P(L_S + L_{SB})\rho_R + \int_0^t dt' PL_{SB} e^{-(1-P)(L_S + L_B)t'} (1-P)L_{SB}\rho_R(t')$$

Back-propagation-trick + relabelling

$$\rho_R(t') = e^{L_S(t'-t)}\rho_R(t) + O(L_{SB}^2)$$

$$\dot{\rho}_R(t) = P(L_S + L_{SB})\rho_R + \int_0^t dt' PL_{SB} e^{-(1-P)(L_S + L_B)t'} (1-P)L_{SB}\rho_R(t)$$

**Approximation 2:** Markov approximation: Short memory

$$K(t) = PL_{SB} e^{-(1-P)(L_S + L_B)t} (1-P) \quad \text{for } t > t_c \quad H_{SB} t_c = 1$$

Motional narrowing limit:  $\alpha < \omega_c / \omega$

Send upper bound of integration to  $\infty$

# Bloch-Redfield equation

What does this superoperator expression mean?

$$\dot{\rho}_R(t) = P(L_S + L_{SB})\rho_R + \int_0^\infty dt' PL_{SB} e^{-(1-P)(L_S + L_B)t'} (1-P)L_{SB}\rho_R(t)$$

$$\dot{\rho}_R(t) = [H_{eff}, \rho_R] + \int_0^\infty dt' Tr_B \left( \left[ H_{SB}(t'), [H_{SB}(0), \rho_R(t)] \right] \right)$$

Double commutator structure! Compact notation:

$$\dot{\rho}_{nm} = -i\omega_{nm}\rho_{nm} + \sum_{kl} R_{nmkl}\rho_{kl} \quad \text{Bloch-Redfield equation}$$

Redfield relaxation tensor:

$$\text{Golden rule rates: } R_{nmkl} = \Gamma_{lmnk} + \Gamma_{lmnk}^- - \delta_{ml} \sum \Gamma_{nssk} - \delta_{nk} \sum \Gamma_{lssm}^-$$

$$\Gamma_{lmnk} = \int_0^\infty dt e^{-i\omega_{nk}t} \left\langle H_{SB,lm}^I(t) H_{SB,nk}^I(0) \right\rangle_B$$

$$\Gamma_{lmnk}^- = \int_0^\infty dt e^{-i\omega_{lm}t} \left\langle H_{SB,lm}^I(0) H_{SB,nk}^I(t) \right\rangle_B$$

# From Redfield to Bloch

Neglect rapidly oscillating terms: Bloch equation

$$M_{z,eq} = \frac{\Gamma_{0110} - \Gamma_{1001}}{\Gamma_{0110} + \Gamma_{1001}} = \tanh\left(\frac{E}{2T}\right) \quad \Gamma_{0110}/\Gamma_{1001} = e^{-E/T} \quad \frac{1}{T_1} = \Gamma_{0110} + \Gamma_{1001}$$

$$\frac{1}{T_2} = \Re\left(\Gamma_{1001} + \Gamma_{0110} + \Gamma_{0000} - \Gamma_{0011} + \Gamma_{1111} - \Gamma_{1100}\right) = \frac{1}{2T_1} + \frac{1}{T_\phi}$$

$$\delta E = \Im\left(\Gamma_{1001} + \Gamma_{0110}\right) \quad \text{(Lamb) Energy shift}$$

Spin-Boson model

$$\frac{1}{T_1} = \frac{\Delta^2}{E^2} S(E)$$

$$\delta E = \frac{\Delta^2}{E^2} P \int_0^\infty d\omega \frac{J(\omega)}{E^2 - \omega^2} = \alpha \frac{\Delta^2}{E^2} \log(E/\omega_c)$$

$$\frac{1}{T_\phi} = \frac{\epsilon^2}{E^2} S(0)$$

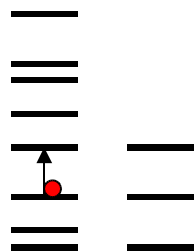
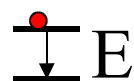
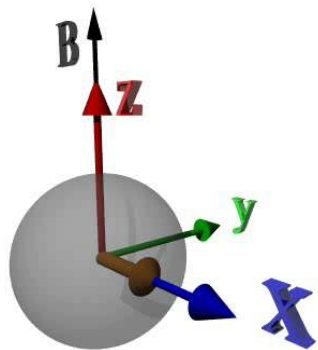
# Bloch-Redfield picture

Generalized master equation + golden rule rates

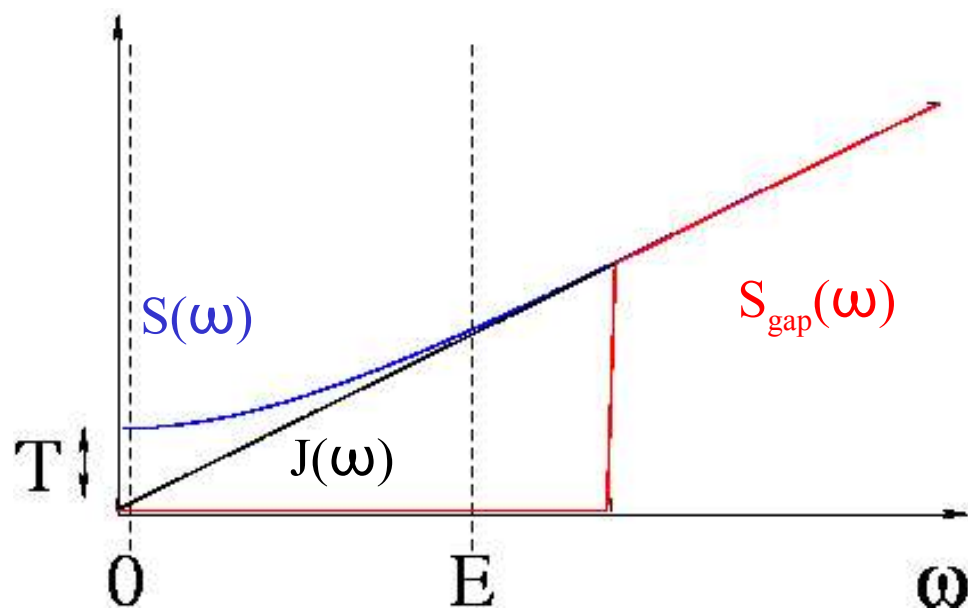
Relaxation

Spin

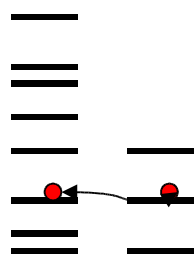
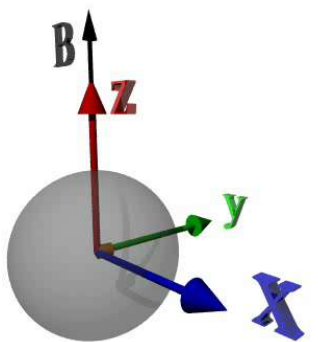
Bath



$$\frac{1}{T_1} = \frac{\Delta^2}{2E^2} S(E)$$



Dephasing



Gapped  $S(\omega)$  = no decoherence?

$$\frac{1}{T_2} = \frac{1}{2T_1} + \frac{\epsilon^2}{2E^2} S(0)$$

# Summary

- Decoherence is irreversible and needs an environment
- Oscillator bath models go a long way
- Obtaining the decoherence model from the electronic circuitry
- QBM and FDT
- Bloch and Bloch-Redfield